

Nonlinear Operator Ideals Between Metric Spaces and Banach Spaces

PART I

MANAF ADNAN SALEH SALEH
 Mathematisches Institut, Universität Jena
 Ernst–Abbe–Platz 2, 07743 Jena, Germany
 E–mail: manaf_math@yahoo.com

Abstract

In this paper we present part I of nonlinear operator ideals theory between metric spaces and Banach spaces. Building upon the definition of operator ideal between arbitrary Banach spaces of A. Pietsch we pose three types of nonlinear versions of operator ideals. We introduce several examples of nonlinear ideals and the relationships between them. For every space ideal \mathbf{A} can be generated by a special nonlinear ideal which consists of those Lipschitz operators admitting a factorization through a Banach space $\mathbf{M} \in \mathbf{A}$. We investigate products and quotients of nonlinear ideals. We devote to constructions three types of new nonlinear ideals from given ones. A “new” is a rule defining nonlinear ideals \mathfrak{A}_{new}^L , \mathfrak{A}_{new}^L , and \mathfrak{A}_{new}^L for every \mathfrak{A} , \mathfrak{A}^L , and \mathfrak{A}^L respectively, are called a Lipschitz procedure. Considering the class of all stable objects for a given Lipschitz procedure we obtain nonlinear ideals having special properties. We present the concept of a (strongly) p –Banach nonlinear ideal ($0 < p < 1$) and prove that the nonlinear ideals of Lipschitz nuclear operators, Lipschitz Hilbert operators, products and quotient are strongly r –Banach nonlinear ideals ($0 < r < 1$).

1 NOTATIONS AND PRELIMINARIES

We introduce concepts and notations that will be used in this article. The letters X , Y and Z will denote pointed metric spaces, i.e. each one has a special point designated by 0. The letters E , F and G will denote Banach spaces. The closed unit ball of a Banach space E is denoted by B_E . The dual space of E is E^* . The class of all bounded linear operators between arbitrary Banach spaces will be denoted by \mathfrak{L} . The symbols \mathbb{R} and \mathbb{N} stand for the set of all real numbers and the set of all natural numbers, respectively. For a Lipschitz mapping T between metric spaces, $Lip(T)$ denotes its Lipschitz constant.

Given metric spaces X and Y , we assume that all Lipschitz functions from X to Y has a special point 0. For the special case $Y = \mathbb{R}$, the Banach space of real–valued Lipschitz functions defined on X that send 0 to 0 with the Lipschitz norm $Lip(\cdot)$ will be denoted by $X^\#$. The space $X^\#$ is called the Lipschitz dual of X .

In this paper, we write \mathbf{A} be a space ideal see [2, Sec. 2.1]. Consider $E_1 \times E_2$ be the Cartesian products of E_1 and E_2 . Put $J_1x_1 := (x_1, 0)$, $J_2x_2 := (0, x_2)$, $Q_1(x_1, x_2) := x_1$, and $Q_2(x_1, x_2) := x_2$.

Recall the Lipschitz dual operator $S^\#$ from $Y^\#$ to $X^\#$ of a map $S \in Lip(X, Y)$ is defined by

$$\langle g, Sx \rangle_{(Y^\#, Y)} = \langle S^\#g, x \rangle_{(X^\#, X)}$$

for every $x \in X$ and $g \in Y^\#$. This is a bounded linear operator and $Lip(S) = \|S^\#\|_{\mathfrak{L}(Y^\#, X^\#)}$, see [6]. The definition of the extension and lifting property of Banach space can be founded in [2, Sec. C.3]. For a Banach space F . Then we put $F^{inj} := \ell_\infty(B_{F^*})$ and $J_F e := (\langle e, e^* \rangle)$ for $e \in F$. Clearly J_F is a metric injection from F into F^{inj} . In this way every Banach space can be identified with a subspace of some Banach space having the metric extension property. For a Banach space E . Then we put $E^{sur} := \ell_1(B_E)$ and $Q_E(\xi_x) := \sum_{B_E} \xi_x x$ for $(\xi_x) \in \ell_1(B_E)$. Clearly Q_E is a metric surjection from E^{sur} onto E . In this way every Banach space can be identified with a quotient space of some Banach space having the metric lifting property.

2 DEFINITIONS AND ELEMENTARY PROPERTIES

Building upon the results of M. G. Cabrera–Padilla, J. A. Chávez–Domínguez, A. Jimenez–Vargas and M. Villegas–Vallecillos [1] we introduce Lipschitz tensor product between pointed metric space X and the topological dual space F^* of a Banach space F .

Similarly to [1, Definition 1.1] we can constructed $X \boxtimes F^*$ as a space of linear functionals on $Lip(X, F)$ spanned by the set $\{\delta_{(x,y)} \boxtimes a : (x, y) \in X \times X; a \in F^*\}$.

The proof of the next lemma is similar to [1, Lemma 1.1] and is therefore omitted.

Lemma 1. *Let $\lambda \in \mathbb{K}$, $(x, y), (x_1, y_1), (x_2, y_2) \in X \times X$ and $a, a_1, a_2 \in F^*$.*

- $\lambda (\delta_{(x,y)} \boxtimes a) = ((\lambda \delta_{(x,y)}) \boxtimes a) = (\delta_{(x,y)} \boxtimes (\lambda a)).$
- $(\delta_{(x_1, y_1)} + \delta_{(x_2, y_2)}) \boxtimes a = \delta_{(x_1, y_1)} \boxtimes a + \delta_{(x_2, y_2)} \boxtimes a.$
- $\delta_{(x,y)} \boxtimes (a_1 + a_2) = \delta_{(x,y)} \boxtimes a_1 + \delta_{(x,y)} \boxtimes a_2.$
- $(\delta_{(x,y)} \boxtimes a) = \delta_{(x,y)} \boxtimes \theta = \theta.$

Remark 1. *We say that $\delta_{(x,y)} \boxtimes a$ is an elementary Lipschitz tensor element in $X \boxtimes F^*$. Note that each element v in $X \boxtimes F^*$ is of the form $v = \sum_{j=1}^m \lambda_j (\delta_{(x_j, y_j)} \boxtimes a_j)$, where $n \in \mathbb{K}$, $\lambda_j \in \mathbb{K}$, $(x_j, y_j) \in X \times X$ and $a_j \in F^*$. This representation of v is not unique. It is worth noting that each element v of $X \boxtimes F^*$ can be represented as $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j)$ since $\lambda (\delta_{(x,y)} \boxtimes a) = (\delta_{(x,y)} \boxtimes (\lambda a))$. This representation of v admits the following refinement.*

The proof of the next lemma is similar to [1, Lemma 1.1] and is therefore omitted.

Lemma 2. Every nonzero Lipschitz tensor $v \in X \boxtimes F^*$ has a representation in the form $\sum_{j=1}^m (\delta_{(z_j, 0)} \boxtimes q_j)$, where

$$m = \min \left\{ k \in \mathbb{N} : \exists z_1, \dots, z_k, q_1, \dots, q_k \in F^* \mid v = \sum_{j=1}^k (\delta_{(z_j, 0)} \boxtimes q_j) \right\}$$

and the points z_1, \dots, z_m in X are distinct from the base point 0 of X and pairwise distinct.

We can concatenate the representations of two elements of $X \boxtimes F^*$ to get a representation of their sum.

Lemma 3. Let $v_1, v_2 \in X \boxtimes F^*$ and let $\sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j)$ and $\sum_{j=1}^{m'} (\delta_{(x'_j, y'_j)} \boxtimes a'_j)$ be representations of v_1 and v_2 , respectively. Then $\sum_{j=1}^{m+m'} (\delta_{(x''_j, y''_j)} \boxtimes a''_j)$, where

$$(x''_j, y''_j) = \begin{cases} (x_j, y_j) & \text{if } j = 1, \dots, m \\ (x'_{j-m}, y'_{j-m}) & \text{if } j = m+1, \dots, m+m' \end{cases}$$

$$a''_j = \begin{cases} a_j & \text{if } j = 1, \dots, m \\ a'_{j-m} & \text{if } j = m+1, \dots, m+m' \end{cases}$$

is a representation of $v_1 + v_2$.

We now describe the action of a Lipschitz tensor $v \in X \boxtimes F^*$ on a function $f \in Lip(X, F)$.

The proof of the next lemma is similar to [1, Lemma 1.4] and is therefore omitted.

Auto **Lemma 4.** Let $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j) \in X \boxtimes F^*$ and $f \in Lip(X, F)$. Then

$$v(f) = \sum_{j=1}^m \langle f x_j - f y_j, a_j \rangle.$$

Our next aim is to characterize the zero Lipschitz tensor. For it we need the following Lipschitz operators.

The proof of the next lemma is similar to [1, Lemma 1.5] and is therefore omitted.

Auto1 **Lemma 5.** Let $g \in X^\#$ and $e \in F$. The map $g \cdot e : X \rightarrow F$, given by

$$(g \cdot e)(x) = g(x) \cdot e,$$

belongs to $Lip(X, F)$ and $Lip(g \cdot e) = Lip(g) \cdot \|e\|$.

The proof of the next proposition is similar to [1, Proposition 1.6] and is therefore omitted.

Proposition 1. If $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j) \in X \boxtimes F^*$, then the following assertions are equivalent:

- $v = 0$.
- $\sum_{j=1}^m (gx_j - gy_j) \cdot \langle a_j, e \rangle = 0$, for every $g \in B_{X^\#}$ and $e \in B_E$.
- $\sum_{j=1}^m (gx_j - gy_j) \cdot a_j = 0$, for every $g \in B_{X^\#}$.

The proof of the next theorem is similar to [1, Theorem 1.7] and is therefore omitted.

Theorem 1. $\langle X \boxtimes F^*, \text{Lip}(X, F) \rangle$ forms a dual pair, where the bilinear form $\langle \cdot, \cdot \rangle$ associated to the dual pair is given, for $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j) \in X \boxtimes F^*$ and $f \in \text{Lip}(X, F)$, by

$$\langle v, f \rangle = \sum_{j=1}^m \langle fx_j - fy_j, a_j \rangle.$$

Since $\langle X \boxtimes F^*, \text{Lip}(X, F) \rangle$ is a dual pair, $\text{Lip}(X, F)$ can be identified with a linear subspace of $(X \boxtimes F^*)'$ as follows.

The proof of the next corollary is similar to [1, Corollary 1.8] and is therefore omitted.

Corollary 1. For every map $f \in \text{Lip}(X, F)$, the functional $\Lambda(f) : X \boxtimes F^* \longrightarrow \mathbb{K}$, given by

$$\Lambda(f)(v) = \sum_{j=1}^m \langle fx_j - fy_j, a_j \rangle$$

for $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j) \in X \boxtimes F^*$, is linear. We say $\Lambda(f)$ is the linear functional on $X \boxtimes F^*$ associated to f . The map $f \mapsto \Lambda(f)$ is a linear monomorphism from $\text{Lip}(X, F)$ into $(X \boxtimes F^*)'$.

Similarly to [1, Definition 2.1] we introduce the concept of Lipschitz tensor product functional of a Lipschitz functional and a bounded linear functional as follows.

Definition 1. Let X be a pointed metric space and F a Banach space. Let $g \in X^\#$ and $a_j \in F^*$. The map $g \boxtimes e : X \boxtimes F^* \longrightarrow \mathbb{K}$, given by

$$(g \boxtimes e)(v) = \sum_{j=1}^m (gx_j - gy_j) \cdot \langle a_j, e \rangle$$

for $v = \sum_{j=1}^m (\delta_{(x_j, y_j)} \boxtimes a_j) \in X \boxtimes F^*$, is called the Lipschitz tensor product functional of g and e .

By Lemma 4 note that

$$(g \boxtimes e)(v) = \sum_{j=1}^m ((g \cdot e)x_j - (g \cdot e)y_j) \cdot \langle a_j, e \rangle = v(g \cdot e).$$

Similarly to [1, Lemma 2.1], the following result which follows easily from this formula gathers some properties of these functionals.

Lemma 6. *Let $g \in X^\#$ and $e \in F$. The functional $(g \boxtimes e) : X \boxtimes F^* \rightarrow \mathbb{K}$ is a well-defined linear map satisfying $\lambda(g \boxtimes e) = (\lambda \cdot g) \boxtimes e = g \boxtimes (\lambda \cdot e)$ for any $\lambda \in \mathbb{K}$. Moreover $(g_1 + g_2) \boxtimes e = g_1 \boxtimes e + g_2 \boxtimes e$ for all $g_1, g_2 \in X^\#$ and $g \boxtimes (e_1 + e_2) = g \boxtimes e_1 + g \boxtimes e_2$ for all $e_1, e_2 \in F$.*

Similarly to [1, Definition 2.2] we introduce the concept of associated Lipschitz tensor product space of $X \boxtimes F^*$.

Definition 2. *Let X be a pointed metric space and F a Banach space. The space $X^\# \boxplus F$ is defined as the linear subspace of $(X \boxtimes F^*)'$ spanned by the set $\{g \boxtimes e : g \in X^\#, e \in F\}$. This space is called the associated Lipschitz tensor product of $X \boxtimes F^*$.*

Similarly to [1, Lemma 2.2] we also derive easily the following fact.

Lemma 7. *For any $\sum_{j=1}^m g_j \boxtimes e_j \in X^\# \boxplus F$ and $\sum_{j=1}^m \delta_{(x_j, y_j)} \boxtimes a_j \in X \boxtimes F^*$, we have*

$$\left(\sum_{j=1}^m g_j \boxtimes e_j \right) \left(\sum_{j=1}^m \delta_{(x_j, y_j)} \boxtimes a_j \right) = \left(\sum_{j=1}^m \delta_{(x_j, y_j)} \boxtimes a_j \right) \left(\sum_{j=1}^m g_j \cdot e_j \right).$$

Each element v^* in $X^\# \boxplus F$ has the form $\sum_{j=1}^m \lambda_j (g_j \boxtimes e_j)$, where $m \in \mathbb{N}$, $\lambda_j \in \mathbb{K}$, $g_j \in X^\#$ and $e_j \in F$ but this representation is not unique. Since $\lambda(g \boxtimes e) = (\lambda g) \boxtimes e = g \boxtimes (\lambda e)$, each element of $X^\# \boxplus F$ can be expressed as $\sum_{j=1}^m g_j \boxtimes e_j$. This representation can be improved as follows.

The proof of the next lemma is similar to [1, Lemma 2.3] and is therefore omitted.

Lemma 8. *Every nonzero element v^* in $X^\# \boxplus F$ has a representation $\sum_{j=1}^m g_j \boxtimes e_j$ such that the functions g_1, \dots, g_m in $X^\#$ are nonzero and the elements e_1, \dots, e_m in F are linearly independent.*

For a pointed metric space X and a Banach space E , we denote by $\mathfrak{L}(X, F)$ the set of all Lipschitz finite rank operators from X to F . Clearly, $Lip_F(X, F)$ is a linear subspace of $Lip(X, F)$. For any $g \in X^\#$ and $e \in F$, we consider in Lemma 5 the elements $g \cdot e$ of $Lip_F(X, F)$ defined by $(g \cdot e)(x) = g(x) \cdot e$ for all $x \in X$. Note that $\text{rank}(g \cdot e) = 1$ if $g \neq 0$ and $e \neq 0$. Now we prove that these elements generate linearly the space $Lip_F(X, F)$.

The proof of the next lemma is similar to [1, Lemma 2.4] and is therefore omitted.

Lemma 9. *Every element $T \in Lip_F(X, F)$ has a representation in the form*

$$T = \sum_{j=1}^m g_j \cdot e_j, \tag{1} \quad \boxed{\text{Auto2}}$$

where $m = \text{rank}(f)$, g_1, \dots, g_m in $X^\#$ and e_1, \dots, e_m in F .

The proof of the next theorem is similar to [1, Theorem 2.5] and is therefore omitted.

Auto11 **Lemma 10.** Let $v^* = \sum_{j=1}^m g_j \boxtimes e_j$ in $X^\# \boxplus F$. The map \tilde{v} from X into F , defined by

$$\tilde{v}(x) = \sum_{j=1}^m g_j(x) \cdot e_j \stackrel{\text{def}}{=} v^*(x),$$

is a linear isomorphism.

Auto9 **Remark 2.** Let $g \in X^\#$ and $e \in F$. Obviously

$$g \boxtimes e : x \longrightarrow \langle x, g \rangle \cdot e$$

is a Lipschitz finite rank operator. Also we have $\text{Lip}(g \boxtimes e) = \text{Lip}(g) \cdot \|e\|$.

Recall that the definition of operator ideal between arbitrary Banach spaces of A. Pietsch [3] and [5] is as follows. Suppose that, for every pair of Banach spaces E and F , we are given a subset $\mathfrak{A}(E, F)$ of $\mathfrak{L}(E, F)$. The class

$$\mathfrak{A} := \bigcup_{E, F} \mathfrak{A}(E, F),$$

Auto5 is said to be an **operator ideal**, or just an **ideal**, if the following conditions are satisfied:

(**OI₀**) $a^* \otimes e \in \mathfrak{A}(E, F)$ for $a^* \in E^*$ and $e \in F$.

(**OI₁**) $S + T \in \mathfrak{A}(E, F)$ for $S, T \in \mathfrak{A}(E, F)$.

(**OI₂**) $BTA \in \mathfrak{A}(E_0, F_0)$ for $A \in \mathfrak{L}(E_0, E)$, $T \in \mathfrak{A}(E, F)$, and $B \in \mathfrak{L}(F, F_0)$.

Condition (**OI₀**) implies that \mathfrak{A} contains nonzero operators.

Building upon the linear version of aforementioned concept "operator ideals" we present three types of nonlinear versions of operator ideals as follow.

The first type is called nonlinear ideal between arbitrary Banach spaces E and F as follow.

Definition 3. Suppose that, for every pair of Banach spaces E and F , we are given a subset $\mathfrak{A}^L(E, F)$ of $\text{Lip}(E, F)$. The class

$$\mathfrak{A}^L := \bigcup_{E, F} \mathfrak{A}^L(E, F),$$

is said to be a **nonlinear operator ideal**, or just a **nonlinear ideal**, if the following conditions are satisfied:

Auto6

(**NOI₀**) $h \boxtimes e \in \mathfrak{A}^L(E, F)$ for $h \in E^\#$ and $e \in F$.

(**NOI₁**) $S + T \in \mathfrak{A}^L(E, F)$ for $S, T \in \mathfrak{A}^L(E, F)$.

(**NOI₂**) $BTA \in \mathfrak{A}^L(E_0, F_0)$ for $A \in \text{Lip}(E_0, E)$, $T \in \mathfrak{A}^L(E, F)$, and $B \in \mathfrak{L}(F, F_0)$.

Condition (**NOI₀**) implies that \mathfrak{A}^L contains nonzero Lipschitz operators.

Remark 3. Suppose that, for every pair of metric spaces X and Banach spaces F , the class

$$Lip := \bigcup_{X,F} Lip(X, F),$$

stands for all Lipschitz maps acting between arbitrary metric spaces and Banach spaces.

The second type is called nonlinear ideal between arbitrary metric spaces X and arbitrary Banach spaces F as follow.

Auto7

Definition 4. Suppose that, for every pair of metric spaces X and Banach spaces F , we are given a subset $\mathfrak{A}^L(X, F)$ of $Lip(X, F)$. The class

$$\mathfrak{A}^L := \bigcup_{X,F} \mathfrak{A}^L(X, F),$$

is said to be a **nonlinear operator ideal**, or just a **nonlinear ideal**, if the following conditions are satisfied:

$$(\widetilde{\text{NOI}}_0) \quad g \boxtimes e \in \mathfrak{A}^L(X, F) \text{ for } g \in X^\# \text{ and } e \in F.$$

$$(\widetilde{\text{NOI}}_1) \quad S + T \in \mathfrak{A}^L(X, F) \text{ for } S, T \in \mathfrak{A}^L(X, F).$$

$$(\widetilde{\text{NOI}}_2) \quad BTA \in \mathfrak{A}^L(X_0, F_0) \text{ for } A \in Lip(X_0, X), T \in \mathfrak{A}^L(X, F), \text{ and } B \in \mathfrak{L}(F, F_0).$$

Condition $(\widetilde{\text{NOI}}_0)$ implies that \mathfrak{A}^L contains nonzero Lipschitz operators.

Remark 4. Suppose that, for every pair of metric spaces X and Y , the class

$$\mathcal{L} := \bigcup_{X,Y} \mathcal{L}(X, Y),$$

stands for all Lipschitz maps acting between arbitrary metric spaces X and Y .

The third type is called nonlinear ideal between arbitrary metric spaces X and Y as follow.

Auto7

Definition 5. Suppose that, for every pair of metric spaces X and Y , we are given a subset $\mathcal{A}^L(X, Y)$ of $\mathcal{L}(X, Y)$. The class

$$\mathcal{A}^L := \bigcup_{X,Y} \mathcal{A}^L(X, Y),$$

is said to be a **nonlinear operator ideal**, or just a **nonlinear ideal**, if the following conditions are satisfied:

$$(\widetilde{\text{NOI}}_0) \quad \mathfrak{A}^L \subset \mathcal{A}^L.$$

$$(\widetilde{\text{NOI}}_1) \quad BTA \in \mathcal{A}^L(X_0, Y_0) \text{ for } A \in \mathcal{L}(X_0, X), T \in \mathcal{A}^L(X, Y), \text{ and } B \in \mathcal{L}(Y, Y_0).$$

Mathematical interpretation of condition $(\widetilde{\mathbf{NOI}_0})$ that the linear space $\mathfrak{A}^L(X, F)$ define as follow.

$$\mathfrak{A}^L(X, F) = \{T \in Lip(X, F) : T \in \mathscr{A}^L(X, F)\},$$

which implies that \mathscr{A}^L contains nonzero Lipschitz operators.

Remark 5. • *It is obvious that the so-called components $\mathfrak{A}^L(X, F)$ of a given nonlinear ideal \mathfrak{A}^L are linear subspaces of the corresponding $Lip(X, F)$ and that $X^\# \boxplus F$ contained in $\mathfrak{A}^L(X, F)$.*

- *In the Definition 3 and Definition 2, for every pair of Banach spaces E and F if we define the linear spaces $\mathfrak{A}(E, F)$, $\mathfrak{A}^L(E, F)$, and $\mathfrak{A}^L(E, F)$, respectively as follow:*

$$\mathfrak{A}^L(E, F) = \{T \in Lip(E, F) : T \in \mathscr{A}^L(E, F)\},$$

$$\mathfrak{A}^L(E, F) = \{T \in Lip(E, F) : T \in \mathfrak{A}^L(E, F)\},$$

$$\mathfrak{A}(E, F) = \{T \in \mathfrak{L}(E, F) : T \in \mathfrak{A}^L(E, F)\},$$

then $\mathfrak{A} \subset \mathfrak{A}^L \subset \mathfrak{A}^L \subseteq \mathscr{A}^L$.

- *Two nonlinear ideals \mathfrak{A}^L \mathfrak{B}^L are called equal, $\mathfrak{A}^L = \mathfrak{B}^L$, if they coincide component-wise. More generally, we shall frequently write $\mathfrak{A}^L \subset \mathfrak{B}^L$ to mean that $\mathfrak{A}^L(X, F) \subset \mathfrak{B}^L(X, F)$ holds for every pair of metric spaces X and Banach spaces F .*

Auto4

Proposition 2. *Let \mathfrak{A}^L be a nonlinear ideal. Then all components $\mathfrak{A}^L(X, F)$ are linear spaces.*

Proof. By $(\widetilde{\mathbf{NOI}_1})$ it remains to show that $T \in \mathfrak{A}^L(X, F)$ and $\lambda \in \mathbb{K}$ imply $\lambda \cdot T \in \mathfrak{A}^L(X, F)$. This follows from $\lambda \cdot T = (\lambda \cdot I_F) \circ T \circ I_X$ and $(\widetilde{\mathbf{NOI}_2})$. ■

Definition 6. *A nonlinear ideal \mathfrak{A}^L is said to be closed if all components $\mathfrak{A}^L(X, F)$ are closed in $Lip(X, F)$ with respect to the Lipschitz operator norm.*

Remark 6. *The adjectives "largest" and "smallest" used below in connection with nonlinear ideals always refer to this "order relation".*

Certain important nonlinear ideals of Lipschitz operators present as follow.

2.1 Largest Nonlinear Ideals

The class $Lip := \bigcup_{X, F} Lip(X, F)$ is the largest nonlinear ideal obviously obtained by considering all Lipschitz operators between metric spaces and Banach spaces.

2.2 Lipschitz Finite Operators (Smallest Nonlinear Ideal)

We write formula (1) in the language of Lipschitz tensor products. A Lipschitz operator $T \in Lip(X, F)$ is finite if it can be written in the form

$$T = \sum_{j=1}^m g_j \boxtimes e_j, \quad (2) \quad \boxed{\text{Auto3}}$$

where g_1, \dots, g_m in $X^\#$ and e_1, \dots, e_m in F .

The class of all Lipschitz finite rank operators is denoted by \mathfrak{F}^L .

Lemma 11. \mathfrak{F}^L is a smallest nonlinear ideal.

Proof. By Lemma 10, we have $X^\# \boxplus F = \mathfrak{F}^L(X, F)$ for all metric spaces X and Banach spaces F . Since $X^\# \boxplus F \subset \mathfrak{A}^L(X, F)$ holds for every nonlinear ideal \mathfrak{A}^L . This proves $\mathfrak{F}^L \subseteq \mathfrak{A}^L$. The verification of the nonlinear ideal properties is following.

The condition $(\widetilde{\mathbf{NOI}_0})$ is hold. To prove the condition $(\widetilde{\mathbf{NOI}_1})$, let T and S in $\mathfrak{F}^L(X, F)$. Then $\sum_{j=1}^m g_j \cdot e_j$ and $\sum_{j=1}^n g'_j \cdot e'_j$ be representation of T and S , respectively. Then $\sum_{j=1}^{m+n} g''_j \cdot e''_j$, where

$$g''_j = \begin{cases} g_j & \text{if } j = 1, \dots, m \\ g'_{j-m} & \text{if } j = m+1, \dots, m+n \end{cases}$$

$$e''_j = \begin{cases} e_j & \text{if } j = 1, \dots, m \\ e'_{j-m} & \text{if } j = m+1, \dots, m+n \end{cases}$$

is a representation of $T + S$. Since by the definition, both $g : X \rightarrow \mathbb{K}$ and $A : X_0 \rightarrow X$ are Lipschitz maps with $A(0) = 0$, then $g \circ A : X_0 \rightarrow \mathbb{K}$ is a Lipschitz map and $g \circ A \in X_0^\#$. Let $h = g \circ A$, $A \in Lip(X_0, X)$, $T \in \mathfrak{F}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$, to show that $BTA \in \mathfrak{F}^L(X_0, F_0)$.

$$\begin{aligned} (BTA)(x_0) &= \left[B \left(\sum_{j=1}^m g_j \cdot e_j \right) A \right] (x_0) = \left[B \left(\sum_{j=1}^m g_j \cdot e_j \right) \right] A(x_0) = B \left[\left(\sum_{j=1}^m g_j \cdot e_j \right) A(x_0) \right] \\ &= B \left[\left(\sum_{j=1}^m g_j(Ax_0) \cdot e_j \right) \right] = \sum_{j=1}^m g_j(Ax_0) \cdot Be_j \\ &= \sum_{j=1}^m (g_j \circ A) x_0 \cdot Be_j = \sum_{j=1}^m h_j(x_0) \cdot Be_j \\ &= \sum_{j=1}^m (h_j \cdot Be_j)(x_0). \end{aligned}$$

Hence $\sum_{j=1}^m h_j \cdot Be_j$ be a representation of BTA , the condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

2.3 Lipschitz Approximable Operators

Recall that the definition of Lipschitz approximable operators of A. Jiménez–Vargas, J. M. Sepulcre, and Moisés Villegas–Vallecillos, [4] is as follows. A Lipschitz operator $T \in Lip(X, F)$ is called Lipschitz approximable if there are $T_1, T_2, T_3, \dots \in \mathfrak{F}^L(X, F)$ with $\lim_n Lip(T - T_n) = 0$.

The class of all Lipschitz approximable operators is denoted by \mathfrak{G}^L .

Lemma 12. \mathfrak{G}^L is a nonlinear ideal.

Proof. The Lipschitz tensor product $g \boxtimes e \in \mathfrak{G}^L(X, F)$. Since there is a sequence $(T_n)_{n \in \mathbb{N}} = g \boxtimes e$ in $\mathfrak{F}^L(X, F)$ with $\lim_n Lip(g \boxtimes e - T_n) = 0$. To prove the condition $(\widetilde{\mathbf{NOI}}_1)$, let T and S in $\mathfrak{G}^L(X, F)$. Then there are sequences $(T_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ in $\mathfrak{F}^L(X, F)$ with $\lim_n Lip(T - T_n) = \lim_n Lip(S - S_n) = 0$. From Proposition 2 there is a sequence $((T_n + S_n))_{n \in \mathbb{N}}$ in $\mathfrak{F}^L(X, F)$ such that $\lim_n Lip(T + S - (T_n + S_n)) = 0$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{G}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Then there is a sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathfrak{F}^L(X, F)$ with $\lim_n Lip(T - T_n) = 0$. To show that $BT A \in \mathfrak{G}^L(X_0, F_0)$, by using the nonlinear composition ideal property there is a sequence $((BT_n A))_{n \in \mathbb{N}}$ in $\mathfrak{F}^L(X_0, F_0)$ such that $\lim_n Lip(BT A - BT_n A) = 0$, the condition $(\widetilde{\mathbf{NOI}}_2)$ satisfied. ■

2.4 Lipschitz Compact Operators

Recall that the definition of Lipschitz compact operators of A. Jiménez–Vargas, J. M. Sepulcre, and Moisés Villegas–Vallecillos, [4] is as follows. A Lipschitz operator $T \in Lip(X, F)$ is called Lipschitz compact if its Lipschitz image is relatively compact in F .

The class of all Lipschitz compact operators is denoted by \mathfrak{K}^L . By [4] the following result is evident.

Lemma 13. \mathfrak{K}^L is a nonlinear ideal.

2.5 Lipschitz Weakly Compact Operators

Recall that the definition of Lipschitz weakly compact operators of A. Jiménez–Vargas, J. M. Sepulcre, and Moisés Villegas–Vallecillos, [4] is as follows. A Lipschitz operator $T \in Lip(X, F)$ is called Lipschitz weakly compact if its Lipschitz image is relatively weakly compact in F .

The class of all Lipschitz compact operators is denoted by \mathfrak{W}^L . By [4] the following result is evident.

Lemma 14. \mathfrak{W}^L is a nonlinear ideal.

2.6 Order Relations between Nonlinear Ideals

The order relations are collected in the following chain of inclusions.

$$smallest := \mathfrak{F}^L \subset \mathfrak{G}^L \subset \mathfrak{K}^L \subset \mathfrak{W}^L \subset Lip =: largest.$$

Auto17

2.7 A-Factorable Lipschitz Operators

Let \mathbf{A} be a space ideal. A Lipschitz operator $T \in Lip(X, F)$ is called \mathbf{A} -factorable if there exists a factorization $T = BA$ such that $A \in Lip(X, \mathbf{M})$, $B \in \mathfrak{L}(\mathbf{M}, F)$, and $\mathbf{M} \in \mathbf{A}$.

The class of all \mathbf{A} -factorable Lipschitz Operators is denoted by $Op^L(\mathbf{A})$.

Proposition 3. $Op^L(\mathbf{A})$ is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}_0})$ satisfied. Since $\mathbb{K} \in \mathbf{A}$, we have $g \boxtimes e = (1 \otimes e) \circ (g \boxtimes 1) \in Op^L_{(X, F)}(\mathbf{A})$, where $1 \otimes e \in \mathfrak{L}(\mathbb{K}, F)$ and $g \boxtimes 1 \in Lip(X, \mathbb{K})$.

To prove the condition $(\widetilde{\mathbf{NOI}_1})$, let $T_i \in Lip(X, F)$ be \mathbf{A} -factorable Lipschitz operators. Then $T_i = B_i A_i$ with $A_i \in Lip(X, \mathbf{M}_i)$, $B_i \in \mathfrak{L}(\mathbf{M}_i, F)$, and $\mathbf{M}_i \in \mathbf{A}$.

$$T_1 + T_2 = (B_1 \circ Q_1 + B_2 \circ Q_2)(J_1 \circ A_1 + J_2 \circ A_2).$$

Hence $T_1 + T_2$ factors through $\mathbf{M}_1 \times \mathbf{M}_2 \in \mathbf{A}$. This proves that $T_1 + T_2 \in Op^L_{(X, F)}(\mathbf{A})$.

Let $A \in Lip(X_0, X)$, $T \in Op^L(\mathbf{A})$, and $B \in \mathfrak{L}(F, F_0)$. Then T admits a factorization

$$T : X \xrightarrow{\tilde{A}} \mathbf{M} \xrightarrow{\tilde{B}} F,$$

where $\tilde{B} \in \mathfrak{L}(\mathbf{M}, F)$ and $\tilde{A} \in Lip(X, \mathbf{M})$. To show that $BTA \in Op^L$. We obtain $B \circ \tilde{B} \in \mathfrak{L}(\mathbf{M}, F_0)$ and $\tilde{A} \circ A \in Lip(X_0, \mathbf{M})$. Hence the Lipschitz operator BTA admits a factorization

$$BTA : X_0 \xrightarrow{\tilde{\tilde{A}}} \mathbf{M} \xrightarrow{\tilde{\tilde{B}}} F_0,$$

where $\tilde{\tilde{B}} = B \circ \tilde{B}$ and $\tilde{\tilde{A}} = \tilde{A} \circ A$, hence the condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

2.8 Products of Nonlinear Ideals

Let \mathfrak{A} be an ideal and \mathfrak{A}^L be nonlinear ideals. A Lipschitz operator $T \in Lip(X, F)$ belongs to the product $\mathfrak{A} \circ \mathfrak{A}^L$ if there is a factorization $T = B \circ A$ with $B \in \mathfrak{A}(M, F)$ and $A \in \mathfrak{A}^L(X, M)$. Here M is a suitable Banach space.

Auto8

Proposition 4. $\mathfrak{A} \circ \mathfrak{A}^L$ is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}_0})$ satisfied. Since the elementary Lipschitz tensor $g \boxtimes e$ admits a factorization

$$g \boxtimes e : X \xrightarrow{g \boxtimes 1} \mathbb{K} \xrightarrow{1 \otimes e} F,$$

where $1 \otimes e \in \mathfrak{A}(\mathbb{K}, F)$ and $g \boxtimes 1 \in \mathfrak{A}^L(X, \mathbb{K})$.

To prove the condition $(\widetilde{\mathbf{NOI}_1})$, let $T_i \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$. Then $T_i = B_i \circ A_i$ with $B_i \in \mathfrak{A}(M_i, F)$ and $A_i \in \mathfrak{A}^L(X, M_i)$. Put $B := B_1 \circ Q_1 + B_2 \circ Q_2$, $A := J_1 \circ A_1 + J_2 \circ A_2$, and $M := M_1 \times M_2$. Now $T_1 + T_2 = B \circ A$, $B \in \mathfrak{A}(M, F)$ and $A \in \mathfrak{A}^L(X, M)$ imply $T_1 + T_2 \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Then T admits a factorization

$$T : X \xrightarrow{\tilde{A}} M \xrightarrow{\tilde{B}} F,$$

where $\tilde{B} \in \mathfrak{A}(M, F)$ and $\tilde{A} \in \mathfrak{A}^L(X, M)$. To show that $BTA \in \mathfrak{A} \circ \mathfrak{A}^L(X_0, F_0)$. By using non-linear composition ideal properties, we obtain $B \circ \tilde{B} \in \mathfrak{A}(M, F_0)$ and $\tilde{A} \circ A \in \mathfrak{A}^L(X_0, M)$. Hence the Lipschitz operator BTA admits a factorization

$$BTA : X_0 \xrightarrow{\tilde{A}} M \xrightarrow{\tilde{B}} F_0,$$

where $\tilde{\tilde{B}} = B \circ \tilde{B}$ and $\tilde{\tilde{A}} = \tilde{A} \circ A$, hence the condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

We raise the following problem which we think is interesting.

Open Problem 1. *Is it true that $\mathfrak{B} \circ \mathfrak{B}^L = \mathfrak{A}^L$?*

2.9 Quotients of Nonlinear Ideals

Let \mathfrak{A} be an operator ideal and \mathfrak{A}^L be a nonlinear ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the quotient $\mathfrak{A}^{-1} \circ \mathfrak{A}^L$ if $B \circ T \in \mathfrak{A}^L(X, F_0)$ for all $B \in \mathfrak{A}(F, F_0)$, where F_0 is an arbitrary Banach space.

Remark 7. *The single symbol \mathfrak{A}^{-1} is without any meaning.*

Auto13

Proposition 5. $\mathfrak{A}^{-1} \circ \mathfrak{A}^L$ is a nonlinear ideal.

Proof. Let $g \in X^\#$, $e \in F$ and B be an arbitrary operator in $\mathfrak{A}(F, F_0)$. Since $g \boxtimes e \in \mathfrak{A}^L(X, F)$ and from nonlinear composition ideal property, we have $B \circ (g \boxtimes e) \in \mathfrak{A}^L(X, F_0)$, hence $g \boxtimes e \in \mathfrak{A}^L$.

Let $T_i \in \mathfrak{Y}^L$ and let B be an arbitrary operator in $\mathfrak{A}(F, F_0)$. To prove the condition $(\widetilde{\mathbf{NOI}_1})$, i.e. to show that $B \circ (T_1 + T_2) \in \mathfrak{A}^L(X, F_0)$. Let x be an arbitrary element in X , from the linearity of the operator B and Proposition 2, we have

$$\begin{aligned} [B \circ (T_1 + T_2)](x) &= B[(T_1 + T_2)(x)] \\ &= B[T_1(x) + T_2(x)] \\ &= B(T_1x) + B(T_2x) \\ &= (B \circ T_1)(x) + (B \circ T_2)(x). \end{aligned}$$

Hence $B \circ (T_1 + T_2) = B \circ T_1 + B \circ T_2$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{Y}^L(X, F)$, and $B \in \mathfrak{A}(F, F_0)$. Then $\tilde{B} \circ T \in \mathfrak{A}^L(X, G)$ for all $\tilde{B} \in \mathfrak{A}(F, G)$, where G is an arbitrary Banach space. To show that $BTA \in \mathfrak{Y}^L(X_0, F_0)$. Let $\tilde{\tilde{B}}$ be an arbitrary operator in $\mathfrak{A}(F_0, G)$, by using the non-linear composition ideal property, and the aforementioned assumption we have $\tilde{\tilde{B}} \circ B \in \mathfrak{A}(F, G)$, $(\tilde{\tilde{B}} \circ B) \circ T \in \mathfrak{A}^L(X, G)$ and $\left[(\tilde{\tilde{B}} \circ B) \circ T \right] \circ A \in \mathfrak{A}^L(X_0, G)$. Hence $\tilde{\tilde{B}} \circ (BTA) \in \mathfrak{A}^L(X_0, G)$. The condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

Remark 8. • A Lipschitz operator belonging to the nonlinear ideal $\mathfrak{B}^{-1} \circ \mathfrak{A}^L$ should be called a **Rosenthal Lipschitz operator**.

- A Lipschitz operator belonging to the nonlinear ideal $\mathfrak{X}^{-1} \circ \mathfrak{A}^L$ should be called a **Grothendieck Lipschitz operator**.

3 Operator Ideals with Special Properties

3.1 Lipschitz Procedures

A rule

$$new : \mathfrak{A} \longrightarrow \mathfrak{A}_{new}^L$$

which defines a new nonlinear ideal \mathfrak{A}_{new}^L for every ideal \mathfrak{A} is called a **semi-Lipschitz procedure**.

A rule

$$new : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{new}^L$$

which defines a new nonlinear ideal \mathfrak{A}_{new}^L for every nonlinear ideal \mathfrak{A}^L is called a **Lipschitz procedure**.

A rule

$$new : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{new}^L$$

which defines a new nonlinear ideal \mathfrak{A}_{new}^L for every nonlinear ideal \mathfrak{A}^L is called a **Lipschitz procedure**.

Remark 9. We now list the following special property:

(M) If $\mathfrak{A}^L \subseteq \mathfrak{B}^L$, then $\mathfrak{A}_{new}^L \subseteq \mathfrak{B}_{new}^L$ (**strong monotony**).

(M') If $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}_{new}^L \subseteq \mathfrak{B}_{new}^L$ (**monotony**).

(I) $(\mathfrak{A}_{new}^L)_{new} = \mathfrak{A}_{new}^L$ for all \mathfrak{A}^L (**idempotence**).

A strong monotone and idempotent Lipschitz procedure is called a **hull Lipschitz procedure** if $\mathfrak{A}^L \subseteq \mathfrak{A}_{new}^L$.

3.2 Closed Nonlinear Ideals

Let \mathfrak{A}^L be a nonlinear ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the closure \mathfrak{A}_{clos}^L if there are $T_1, T_2, T_3, \dots \in \mathfrak{A}^L(X, F)$ with $\lim_n Lip(T - T_n) = 0$.

Proposition 6. \mathfrak{A}_{clos}^L is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}_0})$ satisfied. Since there is a sequence $(T_n)_{n \in \mathbb{N}} = g \boxtimes e$ in $\mathfrak{A}^L(X, F)$ with $\lim_n Lip(g \boxtimes e - T_n) = 0$. To prove the condition $(\widetilde{\mathbf{NOI}_1})$, let T and S in $\mathfrak{A}_{clos}^L(X, F)$. Then there are sequences $(T_n)_{n \in \mathbb{N}}$ and $(S_n)_{n \in \mathbb{N}}$ in $\mathfrak{A}^L(X, F)$ with $\lim_n Lip(T - T_n) = \lim_n Lip(S - S_n) = 0$. From Proposition 2 there is a sequence $((T_n + S_n))_{n \in \mathbb{N}}$ in $\mathfrak{A}^L(X, F)$ such that $\lim_n Lip(T + S - (T_n + S_n)) = 0$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{A}_{clos}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Then there is a sequence $(T_n)_{n \in \mathbb{N}}$ in $\mathfrak{A}^L(X, F)$ with $\lim_n Lip(T - T_n) = 0$. To show that $BT A \in \mathfrak{A}_{clos}^L(X_0, F_0)$, by using the nonlinear composition ideal property there is a sequence $((BT_n A))_{n \in \mathbb{N}}$ in $\mathfrak{A}^L(X_0, F_0)$ such that $\lim_n Lip(BT A - BT_n A) = 0$, the condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

The following statement is evident.

Proposition 7. *The rule*

$$clos : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{clos}^L$$

is a hull Lipschitz procedure.

Remark 10. *The nonlinear ideal \mathfrak{A}^L is called closed if $\mathfrak{A}^L = \mathfrak{A}_{clos}^L$.*

Lemma 15. *\mathfrak{G}^L is the smallest closed nonlinear ideals.*

Proof. By the definition of Lipschitz approximable operator in (2.3) we have $\mathfrak{G}^L := \mathfrak{F}_{clos}^L$. ■

Auto18

Lemma 16. *Let \mathfrak{A}^L be a closed nonlinear ideal. Then $\mathfrak{G}^L \subseteq \mathfrak{A}^L$.*

Proof. Suppose $T \in \mathfrak{G}^L$, hence $T \in \mathfrak{A}_{clos}^L$. Since \mathfrak{A}^L is closed we have $T \in \mathfrak{A}^L$. ■

3.3 Radical

Let \mathfrak{A}^L be a nonlinear operator. A Lipschitz operator $T \in Lip(E, F)$ belongs to the radical \mathfrak{A}_{rad}^L if for every $L \in \mathfrak{L}(F, E)$ there exist $U \in \mathfrak{L}(E)$ and $S \in \mathfrak{A}^L(E)$ such that

$$U(I_E - LT) = I_E - S. \quad (3)$$

Auto10

Proposition 8. *\mathfrak{A}_{rad}^L is a nonlinear ideal.*

Proof. The condition **(NOI₀)** satisfied. Since for arbitrary operator L in $\mathfrak{L}(F, E)$, put $U := I_E$ and $S := L \circ (h \boxtimes e)$. Then $S \in \mathfrak{A}^L(E)$ such that (3) fulfilled. To prove the condition **(NOI₁)**, let T_1 and T_2 in $\mathfrak{A}_{rad}^L(E, F)$. Then, given $L \in \mathfrak{L}(F, E)$, there are $U_1 \in \mathfrak{L}(E)$ and $S_1 \in \mathfrak{A}^L(E)$ with $U_1(I_E - LT_1) = I_E - S_1$. We now choose $U_2 \in \mathfrak{L}(E)$ and $S_2 \in \mathfrak{A}^L(E)$ such that $U_2(I_E - U_1LT_2) = I_E - S_2$. Then

$$U_2U_1[I_E - L(T_1 + T_2)] = U_2[I_E - S_1 - U_1LT_2] = I_E - S_2 - U_2S_1.$$

Since $S_2 + U_2S_1 \in \mathfrak{A}^L(E)$, we have $T_1 + T_2 \in \mathfrak{A}_{rad}^L(E, F)$.

Let $A \in Lip(E_0, E)$, $T \in \mathfrak{A}_{rad}^L(E, F)$, and $B \in \mathfrak{L}(F, F_0)$. Given $L \in \mathfrak{L}(F_0, E_0)$, there exist $U \in \mathfrak{L}(E)$ and $S \in \mathfrak{A}^L(E)$ with $U(I_E - ALBT) = I_E - S$. Define the operators $U_0 := I_{E_0} + LBTUA$ and $S_0 := LBTSA$. Clearly $S_0 \in \mathfrak{A}^L(E_0)$ and

$$\begin{aligned} U_0(I_{E_0} - LBT A) &= I_{E_0} - LBT A + LBTU(I_E - ALBT)A \\ &= I_{E_0} - LBT A + LBT(I_E - S)A \\ &= I_{E_0} - LBTSA = I_{E_0} - S_0. \end{aligned}$$

Therefore $BTA \in \mathfrak{A}_{rad}^L(E_0, F_0)$. ■

Proposition 9. *The rule*

$$rad : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{rad}^L$$

is a hull Lipschitz procedure.

Proof. The property **(M)** is obvious. If $T \in \mathfrak{A}^L(E, F)$ and $L \in \mathfrak{L}(F, E)$, put $U := I_E$ and $S := L \circ T$. Then $S \in \mathfrak{A}^L(E)$ such that (3) hold. Consequently $T \in \mathfrak{A}_{rad}^L(E, F)$. This proves that $\mathfrak{A}^L \subseteq \mathfrak{A}_{rad}^L$. To prove property **(I)**, let $T \in Lip(E, F)$ belong to $(\mathfrak{A}_{rad}^L)_{rad}$. Then, given $L \in \mathfrak{L}(F, E)$, there are $U_1 \in \mathfrak{L}(E, E)$ and $S_1 \in \mathfrak{A}_{rad}^L(E)$ such that $U_1(I_E - LT) = I_E - S_1$. We can now find $U_2 \in \mathfrak{L}(E, E)$ and $S_2 \in \mathfrak{A}^L(E)$ with $U_2(I_E - S_1) = I_E - S_2$. Consequently

$$U_2 U_1 (I_E - LT) = U_2 (I_E - S_1) = I_E - S_2.$$

This implies that $T \in \mathfrak{A}_{rad}^L$. Therefore $(\mathfrak{A}_{rad}^L)_{rad} \subseteq \mathfrak{A}_{rad}^L$. The converse inclusion is trivial. ■

Finally, we show that the symmetry in the definition of \mathfrak{A}^L can be removed.

Auto24

Lemma 17. *Let $T \in \mathfrak{A}_{rad}^L(E, F)$. Then for every $L \in \mathfrak{L}(F, E)$ there exist $U \in \mathfrak{L}(E)$ and $S_1, S_2 \in \mathfrak{A}^L(E)$ such that*

$$U(I_E - LT) = I_E - S_1 \quad \text{and} \quad (I_E - LT)U = I_E - S_2.$$

Proof. We can find $U \in \mathfrak{L}(E)$ and $S_1 \in \mathfrak{A}^L(E)$ with $U(I_E - LT) = I_E - S_1$. Since

$$R := I_E - U = S_1 - ULT \in \mathfrak{A}_{rad}^L(E),$$

there are $U_0 \in \mathfrak{L}(E)$ and $S_0 \in \mathfrak{A}^L(E)$ such that $U_0(I_E - R) = I_E - S_0$. We obtain from $U_0 U + S_0 = I_E$ that

$$\begin{aligned} (I_E - LT)U &= U_0 U (I_E - LT)U + S_0 (I_E - LT)U \\ &= U_0 (I_E - S_1)U + S_0 (I_E - LT)U = I_E - S_2, \end{aligned}$$

where

$$S_2 := S_0 (I_E - U + LTU) + U_0 S_1 U \in \mathfrak{A}^L(E). \quad \text{■}$$

Lemma 18. *Let $T \in \mathfrak{A}_{rad}^L(E, F)$. Then for every $L \in \mathfrak{L}(F, E)$ there exist $V \in \mathfrak{L}(F)$ and $P_1, P_2 \in \mathfrak{A}^L(F)$ such that*

$$V(I_F - TL) = I_F - P_1 \quad \text{and} \quad (I_F - TL)V = I_F - P_2.$$

Proof. Apply Lemma 19 to $TL \in \mathfrak{A}^L(F)$. ■

3.4 Symmetric Nonlinear Ideals

Auto19

Let \mathfrak{A} be an ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the Lipschitz dual ideal \mathfrak{A}_{dual}^L if $T_{|F^*}^\# \in \mathfrak{A}(F^*, X^\#)$.

Auto24

Lemma 19. Let T be in $\mathfrak{L}^L(X, F)$, with $T = \sum_{j=1}^m g_j \boxtimes e_j$. Then $T_{|F^*}^\# = \sum_{j=1}^m \hat{e}_j \otimes g_j$, where $e \mapsto \hat{e}$ is the natural embedding of a space F into its second dual F^{**} .

Proof. We have $Tx = \sum_{j=1}^m g_j(x)e_j$ for $x \in X$. So for $y^* \in F^*$,

$$\left\langle T_{|F^*}^\# y^*, x \right\rangle_{(X^\#, X)} = \langle y^*, Tx \rangle_{(F^*, F)} = \sum_{j=1}^m g_j(x) y^*(e_j).$$

Hence $T_{|F^*}^\# y^* = \sum_{j=1}^m y^*(e_j) g_j$. This proves the statement for $T_{|F^*}^\#$. ■

Auto16

Lemma 20. Let $T, S \in Lip(X, F)$, $A \in Lip(X_0, X)$, and $B \in \mathfrak{L}(F, F_0)$. Then

- $(T + S)_{|F^*}^\# = T_{|F^*}^\# + S_{|F^*}^\#$.
- $(BTA)_{|F_0^*}^\# = A^\# T_{|F^*}^\# B^*$.

Proof. For $y^* \in F^*$ and $x \in X$, we have

$$\begin{aligned} \left\langle (T + S)_{|F^*}^\# y^*, x \right\rangle_{(X^\#, X)} &= \langle y^*, (T + S)x \rangle_{(F^*, F)} \\ &= \langle y^*, Tx + Sx \rangle_{(F^*, F)} \\ &= \langle y^*, Tx \rangle_{(F^*, F)} + \langle y^*, Sx \rangle_{(F^*, F)} \\ &= \left\langle T_{|F^*}^\# y^*, x \right\rangle_{(X^\#, X)} + \left\langle S_{|F^*}^\# y^*, x \right\rangle_{(X^\#, X)} \end{aligned}$$

Hence $(T + S)_{|F^*}^\# = T_{|F^*}^\# + S_{|F^*}^\#$.

For $y_0^* \in F_0^*$ and $x_0 \in X_0$, we have

$$\begin{aligned} \langle y_0^*, BTA(x_0) \rangle_{(F_0^*, F_0)} &= \langle y_0^*, B(TAx_0) \rangle_{(F_0^*, F_0)} \\ &= \langle B^* y_0^*, T(Ax_0) \rangle_{(F^*, F)} \\ &= \left\langle T_{|F^*}^\# B^* y_0^*, Ax_0 \right\rangle_{(X^\#, X)} \\ &= \left\langle A^\# T_{|F^*}^\# B^* y_0^*, x_0 \right\rangle_{(X_0^\#, X_0)} \end{aligned}$$

But also $\langle y_0^*, BTA(x) \rangle_{(F_0^*, F_0)} = \left\langle (BTA)_{|F_0^*}^\# y_0^*, x_0 \right\rangle_{(X_0^\#, X_0)}$. Therefore $(BTA)_{|F_0^*}^\# = A^\# T_{|F^*}^\# B^*$. ■

Proposition 10. \mathfrak{A}_{dual}^L is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}_0})$ satisfied. From Lemma 19 we obtain $(g \boxtimes e)_{|_{F^*}}^\# = \hat{e} \otimes g \in \mathfrak{A}(F^*, X^\#)$. To prove the condition $(\widetilde{\mathbf{NOI}_1})$, let T and S in $\mathfrak{A}_{dual}^L(X, F)$. Let $T_{|_{F^*}}^\#$ and $S_{|_{F^*}}^\#$ in $\mathfrak{A}(F^*, X^\#)$, from Lemma 20 we have $(T + S)_{|_{F^*}}^\# = T_{|_{F^*}}^\# + S_{|_{F^*}}^\# \in \mathfrak{A}(F^*, X^\#)$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{A}_{dual}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Also from Lemma 20 we have $(BTA)_{|_{F_0^*}}^\# = A^\# T_{|_{F^*}}^\# B^* \in \mathfrak{A}(F_0^*, X_0^\#)$, hence the condition $(\widetilde{\mathbf{NOI}_2})$ satisfied. ■

Proposition 11. The rule

$$dual : \mathfrak{A} \longrightarrow \mathfrak{A}_{dual}^L$$

is a monotone Lipschitz procedure.

Remark 11. A nonlinear ideal \mathfrak{A}^L is called **symmetric** if $\mathfrak{A}^L \subseteq \mathfrak{A}_{dual}^L$. In case $\mathfrak{A}^L = \mathfrak{A}_{dual}^L$ the nonlinear ideal is said to be **completely symmetric**.

Lemma 21. The nonlinear ideal \mathfrak{F}^L is completely symmetric.

Proof. Let $T \in \mathfrak{F}^L(X, F)$, then T can be representation in the form $\sum_{j=1}^m g_j \boxtimes e_j$. From Lemma 19 we have $T_{|_{F^*}}^\# = \sum_{j=1}^m \hat{e}_j \otimes g_j \in F^{**} \otimes X^\# \equiv \mathfrak{F}(F^*, X^\#)$. Hence $T \in \mathfrak{F}_{dual}^L(X, F)$.

Let $T \in \mathfrak{F}_{dual}^L(X, F)$ then $T_{|_{F^*}}^\# \in \mathfrak{F}(F^*, X^\#)$ hence $T_{|_{F^*}}^\#$ can be representation in the form $\sum_{j=1}^m \hat{e}_j \otimes g_j$. For $y^* \in F^*$ and $x \in X$, we have

$$\begin{aligned} \langle y^*, Tx \rangle_{(F^*, F)} &= \left\langle T_{|_{F^*}}^\# y^*, x \right\rangle_{(X^\#, X)} \\ &= \left\langle \sum_{j=1}^m \hat{e}_j \otimes g_j (y^*), x \right\rangle_{(X^\#, X)} \\ &= \left\langle \sum_{j=1}^m \hat{e}_j(y^*) \cdot g_j, x \right\rangle_{(X^\#, X)} \\ &= \left\langle \sum_{j=1}^m y^*(e_j) \cdot g_j, x \right\rangle_{(X^\#, X)} \\ &= \sum_{j=1}^m g_j(x) \cdot y^*(e_j) \\ &= \left\langle y^*, \sum_{j=1}^m g_j \boxtimes e_j (x) \right\rangle_{(F^*, F)}. \end{aligned}$$

Hence $T = \sum_{j=1}^m g_j \boxtimes e_j \in \mathfrak{L}^L(X, F)$. ■

3.5 Regular Nonlinear Ideals

Let \mathfrak{A}^L be a nonlinear ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the regular hull \mathfrak{A}_{reg}^L if $K_F T \in \mathfrak{A}^L(X, F^{**})$.

Proposition 12. \mathfrak{A}_{reg}^L is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}}_0)$ satisfied. Since $g \boxtimes e \in \mathfrak{A}^L(X, F)$ and using nonlinear composition ideal property we have $K_F(g \boxtimes e) \in \mathfrak{A}^L(X, F^{**})$. To prove the condition $(\widetilde{\mathbf{NOI}}_1)$, let T and S in $\mathfrak{A}_{reg}^L(X, F)$. Then $K_F T$ and $K_F S$ in $\mathfrak{A}^L(X, F^{**})$, we have $K_F(T + S) = K_F T + K_F S \in \mathfrak{A}^L(X, F^{**})$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{A}_{reg}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Then

$$\begin{array}{ccccc} X & \xrightarrow{T} & F & \xrightarrow{K_F} & F^{**} \\ \uparrow A & & \downarrow B & & \downarrow B^{**} \\ X_0 & \xrightarrow{BTA} & F_0 & \xrightarrow{K_{F_0}} & F_0^{**} \end{array}$$

Consequently $K_{F_0}(BTA) = B^{**}(K_F T)A \in \mathfrak{A}^L$, hence the condition $(\widetilde{\mathbf{NOI}}_2)$ satisfied. ■

Proposition 13. The rule

$$reg : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{reg}^L$$

is a hull Lipschitz procedure.

Proof. The property (\mathbf{M}) is obvious. From the nonlinear composition ideal we obtain $\mathfrak{A}^L \subseteq \mathfrak{A}_{reg}^L$. To show the idempotence. Let $T \in Lip(X, F)$ belong to $(\mathfrak{A}_{reg}^L)_{reg}$. Then $K_F T \in \mathfrak{A}_{reg}^L(X, F^{**})$ and $K_{F^{**}} K_F T \in \mathfrak{A}^L(X, F^{****})$. Now $I_{F^{**}} = (K_{F^*})^* K_{F^{**}}$ implies

$$K_F T = (K_{F^*})^* (K_{F^{**}} K_F T) \in \mathfrak{A}^L(X, F^{**})$$

and therefore $T \in \mathfrak{A}_{reg}^L(X, F)$. Hence $(\mathfrak{A}_{reg}^L)_{reg} \subseteq \mathfrak{A}_{reg}^L$. The converse inclusion is trivial. ■

Remark 12. A nonlinear ideal \mathfrak{A}^L is called regular if $\mathfrak{A}^L = \mathfrak{A}_{reg}^L$.

Auto25

Proposition 14. Let \mathfrak{A} be an ideal. Then \mathfrak{A}_{dual}^L is regular.

Proof. We check only the inclusion $(\mathfrak{A}_{dual}^L)_{reg} \subseteq \mathfrak{A}_{dual}^L$. Let $T \in Lip(X, F)$ belong to $(\mathfrak{A}_{dual}^L)_{reg}$. Then $K_F T \in \mathfrak{A}_{dual}^L(X, F^{**})$ and $T|_{F^*}^\# (K_F)^* \in \mathfrak{A}(F^{***}, X^\#)$. Now $I_{F^*} = (K_F)^* K_{F^*}$ implies $T|_{F^*}^\# = T|_{F^*}^\# (K_F)^* K_{F^*} \in \mathfrak{A}(F^*, X^\#)$ then $T \in \mathfrak{A}_{dual}^L(X, F)$. ■

The Proposition 14 gives the following result.

Lemma 22. *If \mathfrak{A} be an ideal, then every completely symmetric nonlinear ideal is regular and symmetric.*

Lemma 23. *The nonlinear ideal \mathfrak{F}^L is regular.*

Proof. The regularity of \mathfrak{F}^L is implied by its completely symmetry. ■

3.6 Injective Nonlinear Ideals

Auto20

Let \mathfrak{A}^L be a nonlinear ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the injective hull \mathfrak{A}_{inj}^L if $J_F T \in \mathfrak{A}^L(X, F^{inj})$.

Proposition 15. *\mathfrak{A}_{inj}^L is a nonlinear ideal.*

Proof. The condition $(\widetilde{\mathbf{NOI}}_0)$ satisfied. Since $g \boxtimes e \in \mathfrak{A}^L(X, F)$ and using nonlinear composition ideal property we have $J_F(g \boxtimes e) \in \mathfrak{A}^L(X, F^{inj})$. To prove the condition $(\widetilde{\mathbf{NOI}}_1)$, let T and S in $\mathfrak{A}_{inj}^L(X, F)$. Then $J_F T$ and $J_F S$ in $\mathfrak{A}^L(X, F^{inj})$, we have $J_F(T + S) = J_F T + J_F S \in \mathfrak{A}^L(X, F^{inj})$.

Let $A \in Lip(X_0, X)$, $T \in \mathfrak{A}_{inj}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. Since F_0^{inj} has the extension property, there exists $B^{inj} \in \mathfrak{L}(F^{inj}, F_0^{inj})$ such that

$$\begin{array}{ccccc} X & \xrightarrow{T} & F & \xrightarrow{J_F} & F^{inj} \\ \uparrow A & & \downarrow B & & \downarrow B^{inj} \\ X_0 & \xrightarrow{BTA} & F_0 & \xrightarrow{J_{F_0}} & F_0^{inj} \end{array}$$

Consequently $J_{F_0}(BTA) = B^{inj}(J_F T)A \in \mathfrak{A}^L$, hence the condition $(\widetilde{\mathbf{NOI}}_2)$ satisfied. ■

Auto21

Lemma 24. *Let F be a Banach space possessing the extension property. Then $\mathfrak{A}^L(X, F) = \mathfrak{A}_{inj}^L(X, F)$.*

Proof. By hypothesis there exists $B \in \mathfrak{L}(F^{inj}, F)$ such that $BJ_F = I_F$. Therefore $T \in \mathfrak{A}_{inj}^L(X, F)$ implies that $T = B(J_F T) \in \mathfrak{A}^L(X, F)$. This proves that $\mathfrak{A}_{inj}^L \subseteq \mathfrak{A}^L$. The converse inclusion is obvious. ■

Auto 14

Proposition 16. *The rule*

$$inj : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{inj}^L$$

is a hull Lipschitz procedure.

Proof. The property **(M)** is obvious. From the preceding lemma we obtain $\mathfrak{A}^L \subseteq \mathfrak{A}_{inj}^L$. To show the idempotence. Let $T \in Lip(X, F)$ belong to $(\mathfrak{A}_{inj}^L)_{inj}$. Then $J_F T \in \mathfrak{A}_{inj}^L(X, F^{inj})$, and the preceding lemma implies $J_F T \in \mathfrak{A}^L(X, F^{inj})$. Consequently $T \in \mathfrak{A}_{inj}^L(X, F)$. Thus $(\mathfrak{A}_{inj}^L)_{inj} \subseteq \mathfrak{A}_{inj}^L$. The converse inclusion is trivial. ■

Proposition 17. *Let \mathbf{A} be a space ideal. Then $[Op^L(\mathbf{A})]_{inj} \subseteq Op^L(\mathbf{A}^{inj})$.*

Proof. Let $T \in Lip(X, F)$ belong to $[Op^L(\mathbf{A})]_{inj}$. Then $J_F T \in Op^L(\mathbf{A})(X, F^{inj})$, so $J_F T = BA$ such that $A \in Lip(X, \mathbf{M})$, $B \in \mathfrak{L}(\mathbf{M}, F^{inj})$, and $\mathbf{M} \in \mathbf{A}$. Put $\mathbf{M}_0 := \overline{R_A}$, and let $A_0 \in Lip(X, \mathbf{M}_0)$ be the Lipschitz operator induced by A . Obviously $B\mathbf{m} \in R_{J_F}$ for $\mathbf{m} \in R_A$. Consequently $B(\mathbf{M}_0) \subseteq R_{J_F}$, and $B_0 := J_F^{-1} B J_{\mathbf{M}_0}^{\mathbf{M}}$ is well-defined. Finally, $T = B_0 A_0$ and $\mathbf{M}_0 \in \mathbf{A}^{inj}$ imply $T \in Op^L(\mathbf{A}^{inj})$. This proves $[Op^L(\mathbf{A})]_{inj} \subseteq Op^L(\mathbf{A}^{inj})$. ■

Remark 13. *A nonlinear ideal \mathfrak{A}^L is called injective if $\mathfrak{A}^L = \mathfrak{A}_{inj}^L$.*

The injectivity of a nonlinear ideal \mathfrak{A}^L means that it does not depend on the size of the target space F whether or not a Lipschitz operator $T \in Lip(X, F)$ belongs to \mathfrak{A}^L .

Auto22

Proposition 18. *A nonlinear ideal \mathfrak{A}^L is injective if and only if for every injection $J \in \mathfrak{L}(F_0, F)$ and for every Lipschitz operator $T_0 \in Lip(X, F_0)$ it follows from $JT_0 \in \mathfrak{A}^L(X, F)$ that $T_0 \in \mathfrak{A}^L(X, F_0)$.*

Proof. To check the necessity we consider an injective nonlinear ideal \mathfrak{A}^L . Let $T_0 \in Lip(X, F_0)$ such that $JT_0 \in \mathfrak{A}^L(X, F)$. Since F_0^{inj} has the extension property, there is $B \in \mathfrak{L}(F, F_0^{inj})$ with $J_{F_0} = BJ$. Hence it follows from $J_{F_0} T_0 = B(JT_0) \in \mathfrak{A}^L(X, F_0^{inj})$ that $T_0 \in \mathfrak{A}_{inj}^L(X, F_0) = \mathfrak{A}^L(X, F_0)$.

Conversely, let us suppose that the given condition is satisfied. If $T_0 \in \mathfrak{A}_{inj}^L(X, F_0)$, then $J_{F_0} T_0 \in \mathfrak{A}^L(X, F_0^{inj})$. Since J_{F_0} is an injection, we obtain $T_0 \in \mathfrak{A}^L(X, F_0)$. Therefore $\mathfrak{A}^L = \mathfrak{A}_{inj}^L$, which proves the sufficiency. ■

3.7 Surjective Nonlinear Ideals

Let \mathfrak{A}^L be a nonlinear operator. A Lipschitz operator $T \in Lip(E, F)$ belongs to the surjective hull \mathfrak{A}_{sur}^L if $TQ_E \in \mathfrak{A}^L(E^{sur}, F)$.

Proposition 19. *\mathfrak{A}_{sur}^L is a nonlinear ideal.*

Proof. The condition **(NOI₀)** satisfied. Since $h \boxtimes e \in \mathfrak{A}^L(E, F)$ and using nonlinear composition ideal property we have $(h \boxtimes e) Q_E \in \mathfrak{A}^L(E^{sur}, F)$. To prove the condition **(NOI₁)**, let T_1 and T_2 in $\mathfrak{A}_{sur}^L(E, F)$. Since $(T_1 + T_2) Q_E = T_1 Q_E + T_2 Q_E$, we have $T_1 + T_2 \in \mathfrak{A}_{sur}^L(E, F)$.

Let $A \in Lip(E_0, E)$, $T \in \mathfrak{A}_{sur}^L(E, F)$, and $B \in \mathfrak{L}(F, F_0)$. Since E_0^{sur} has the lifting property, there exists $B^{sur} \in \mathfrak{L}(E_0^{sur}, E^{sur})$ such that

$$\begin{array}{ccccc} E^{sur} & \xrightarrow{Q_E} & E & \xrightarrow{T} & F \\ \uparrow B^{sur} & & \downarrow A & & \downarrow B \\ E_0^{sur} & \xrightarrow{Q_{E_0}} & E_0 & \xrightarrow{BTA} & F_0 \end{array}$$

Consequently $(BTA) Q_{E_0} = B(TQ_E) B^{sur} \in \mathfrak{A}^L(E_0^{sur}, F_0)$, hence the condition **(NOI₂)**. ■

Lemma 25. *Let E be a Banach space with the lifting property. Then $\mathfrak{A}^L(E, F) = \mathfrak{A}_{sur}^L(E, F)$.*

Proof. By hypothesis there exists $B \in \mathfrak{L}(E, E^{sur})$ such that $Q_E B = I_E$. Therefore $T \in \mathfrak{A}_{sur}^L(E, F)$ implies that $T = (TQ_E) B \in \mathfrak{A}^L(E, F)$. This proves that $\mathfrak{A}_{sur}^L \subseteq \mathfrak{A}^L$. The converse inclusion is obvious. ■

Similarly to Proposition 16 and from the preceding lemma we have

Proposition 20. *The rule*

$$sur : \mathfrak{A}^L \longrightarrow \mathfrak{A}_{sur}^L$$

is a hull Lipschitz procedure.

Recall the definition of **A**-factorable Lipschitz Operators in (2.7) and we assume here that $X = E$. It is evident $\mathfrak{A}^L := Op^L(\mathbf{A})$ is a nonlinear operator.

Proposition 21. *Let \mathbf{A} be a space ideal. Then $[Op^L(\mathbf{A})]_{sur} \subseteq Op^L(\mathbf{A}^{sur})$.*

Proof. Let $T \in Lip(E, F)$ belong to $[Op^L(\mathbf{A})]_{sur}$. Then $TQ_E \in Op^L(\mathbf{A})(E^{sur}, F)$, so $TQ_E = BA$ such that $A \in Lip(E^{sur}, \mathbf{M})$, $B \in \mathfrak{L}(\mathbf{M}, F)$, and $\mathbf{M} \in \mathbf{A}$. Put $\mathbf{M}_0 := \overline{D_B}$, and let $B_0 \in \mathfrak{L}(\mathbf{M}_0, F)$ be the bounded linear operator induced by B . Obviously $A((x_j)_j) \in D_B$ for $(x_j)_j \in D_{Q_E}$. Consequently $A(D_{Q_E}) \subseteq \mathbf{M}_0$, and $A_0 := J_{\mathbf{M}_0}^{\mathbf{M}} A Q_E^{-1}$ is well-defined. Finally, $T = B_0 A_0$ and $\mathbf{M}_0 \in \mathbf{A}^{sur}$ imply $T \in Op^L(\mathbf{A}^{sur})$. This proves $[Op^L(\mathbf{A})]_{sur} \subseteq Op^L(\mathbf{A}^{sur})$. ■

Remark 14. *A nonlinear ideal \mathfrak{A}^L is called surjective if $\mathfrak{A}^L = \mathfrak{A}_{sur}^L$.*

The surjectivity of a nonlinear ideal \mathfrak{A}^L means that it does not depend on the size of the source space E whether or not a Lipschitz operator $T \in Lip(E, F)$ belongs to \mathfrak{A}^L .

Proposition 22. *A nonlinear ideal \mathfrak{A}^L is surjective if and only if for every surjection $Q \in \mathfrak{L}(E, E_0)$ and for every Lipschitz operator $T_0 \in Lip(E_0, F)$ it follows from $T_0 Q \in \mathfrak{A}^L(E, F)$ that $T_0 \in \mathfrak{A}^L(E_0, F)$.*

Proof. To check the necessity we consider a surjective nonlinear ideal \mathfrak{A}^L . Let $T_0 \in Lip(E_0, F)$ such that $T_0 Q \in \mathfrak{A}^L(E, F)$. Since E_0^{sur} has the lifting property, there is $B \in \mathfrak{L}(E_0^{sur}, E)$ with $Q_{E_0} = QB$. Hence it follows from $T_0 Q_{E_0} = (T_0 Q) B \in \mathfrak{A}^L(E_0^{sur}, F)$ that $T_0 \in \mathfrak{A}_{sur}^L(E_0, F) = \mathfrak{A}^L(E_0, F)$.

Conversely, let us suppose that the given condition is satisfied. If $T_0 \in \mathfrak{A}_{sur}^L(E_0, F)$, then $T_0 Q_{E_0} \in \mathfrak{A}^L(E_0^{sur}, F)$. Since Q_{E_0} is a surjection, we obtain $T_0 \in \mathfrak{A}^L(E_0, F)$. Therefore $\mathfrak{A}^L = \mathfrak{A}_{sur}^L$, which proves the sufficiency. ■

Recall the definitions of symmetric and injective nonlinear ideal in (3.4) and (3.6), respectively. We assume here that $X = E$. It is evident \mathfrak{A}_{dual}^L and \mathfrak{A}_{inj}^L is a nonlinear ideal.

Lemma 26. $(\mathfrak{A}_{dual}^L)_{inj} \subseteq (\mathfrak{A}_{sur}^L)_{dual}^L$.

Proof. Let $T \in Lip(E, F)$ belong to $(\mathfrak{A}_{dual}^L)_{inj}$. Then $J_F T \in \mathfrak{A}_{dual}^L(E, F^{inj})$ and $T|_{(F^{inj})^*}^\# J_F^* \in \mathfrak{A}((F^{inj})^*, E^\#)$. Since J_F^* is a surjection, it follows from [2, Sec. 4.7.9] that $T|_{(F^{inj})^*}^\# \in \mathfrak{A}_{sur}^L$. Therefore $T \in (\mathfrak{A}_{sur}^L)_{dual}^L$. ■

Lemma 27. *For every surjective ideal the Lipschitz dual ideal is injective.*

Proof. It follows from $\mathfrak{A} = \mathfrak{A}_{sur}$ that $\mathfrak{A}_{dual}^L = (\mathfrak{A}_{sur})_{dual}^L \supseteq (\mathfrak{A}_{dual}^L)_{inj}$. ■

Open Problem 2. *Every Banach space E_{sur} possesses the approximation property. Is it true that $\mathfrak{G}_{sur}^L = \mathfrak{A}^L$?*

3.8 Minimal Nonlinear Ideals

Let \mathfrak{A} be an ideal. A Lipschitz operator $T \in Lip(X, F)$ belongs to the minimal \mathfrak{A}_{min}^L if $T = BT_0A$, where $B \in \mathfrak{G}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $A \in \mathfrak{G}^L(X, G_0)$. In the other words $\mathfrak{A}_{min}^L := \mathfrak{G} \circ \mathfrak{A} \circ \mathfrak{G}^L$.

Proposition 23. \mathfrak{A}_{min}^L is a nonlinear ideal.

Proof. The condition $(\widetilde{\mathbf{NOI}}_0)$ satisfied. Since the elementary Lipschitz tensor $g \boxtimes e$ admits a factorization

$$g \boxtimes e : X \xrightarrow{g \boxtimes 1} \mathbb{K} \xrightarrow{1 \boxtimes 1} \mathbb{K} \xrightarrow{1 \boxtimes e} F,$$

where $1 \boxtimes e \in \mathfrak{G}(\mathbb{K}, F)$, $1 \boxtimes 1 \in \mathfrak{A}(\mathbb{K}, \mathbb{K})$, and $g \boxtimes 1 \in \mathfrak{G}^L(X, \mathbb{K})$. To prove the condition $(\widetilde{\mathbf{NOI}}_1)$, let $T_i \in \mathfrak{G} \circ \mathfrak{A} \circ \mathfrak{G}^L(X, F)$. Then $T_i = B_i T_0^i A_i$, where $B_i \in \mathfrak{G}(F_0^i, F)$, $T_0^i \in \mathfrak{A}(G_0^i, F_0^i)$, and $A_i \in \mathfrak{G}^L(X, G_0^i)$. Put $B := B_1 \circ Q_1 + B_2 \circ Q_2$, $T_0 := \tilde{J}_1 \circ T_0^1 \circ \tilde{Q}_1 + \tilde{J}_2 \circ T_0^2 \circ \tilde{Q}_2$, and $A := J_1 \circ A_1 + J_2 \circ A_2$. Now $T_1 + T_2 = B \circ T_0 \circ A$, $B \in \mathfrak{G}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $A \in \mathfrak{G}^L(X, G_0)$ imply $T_1 + T_2 \in \mathfrak{G} \circ \mathfrak{A} \circ \mathfrak{G}^L(X, F)$.

Let $A \in \text{Lip}(X_0, X)$, $T \in \mathfrak{G} \circ \mathfrak{A} \circ \mathfrak{G}^L(X, F)$, and $B \in \mathfrak{L}(F, R_0)$. Then T admits a factorization

$$T : X \xrightarrow{\tilde{A}} G_0 \xrightarrow{T_0} F_0 \xrightarrow{\tilde{B}} F,$$

where $\tilde{B} \in \mathfrak{G}(F_0, F)$, $T_0 \in \mathfrak{A}(G_0, F_0)$, and $\tilde{A} \in \mathfrak{G}^L(X, G_0)$. To show that $BT A \in \mathfrak{G} \circ \mathfrak{A} \circ \mathfrak{G}^L(X_0, R_0)$. By using the non-linear composition ideal properties, we obtain $B \circ \tilde{B} \in \mathfrak{G}(F_0, R_0)$ and $\tilde{A} \circ A \in \mathfrak{G}^L(X_0, G_0)$. Hence the Lipschitz operator $BT A$ admits a factorization

$$BT A : X_0 \xrightarrow{\tilde{\tilde{A}}} G_0 \xrightarrow{T_0} F_0 \xrightarrow{\tilde{\tilde{B}}} R_0,$$

where $\tilde{\tilde{B}} = B \circ \tilde{B}$ and $\tilde{\tilde{A}} = \tilde{A} \circ A$, hence the condition $(\widetilde{\text{NOI}}_2)$ satisfied. ■

Proposition 24. *The rule*

$$\min : \mathfrak{A} \longrightarrow \mathfrak{A}_{\min}^L$$

is a monotone Lipschitz procedure.

Remark 15. • *It is evident $\mathfrak{A}_{\min}^L \subseteq \mathfrak{G}^L$.*

- *If \mathfrak{A}^L is a closed nonlinear ideal, then $\mathfrak{A}_{\min}^L \subseteq \mathfrak{A}^L$.*
- *A nonlinear ideal \mathfrak{A}^L is called minimal if $\mathfrak{A}_{\min}^L \subseteq \mathfrak{A}^L$.*

Lemma 28. \mathfrak{F}^L *is a minimal nonlinear ideal.*

Proof. Since \mathfrak{F}^L is closed we obtain $\mathfrak{F}_{\min}^L \subseteq \mathfrak{F}^L$. ■

4 Lipschitz p-Normed Nonlinear Ideals

Let \mathfrak{A}^L be a nonlinear ideal. A map \mathbf{A}^L from \mathfrak{A}^L to \mathbb{R}^+ is called a **Lipschitz p-norm** ($0 < p \leq 1$) if the following conditions are satisfied:

$$(\widetilde{\text{QNOI}}_0) \quad \mathbf{A}^L(g \boxtimes e) = \text{Lip}(g) \cdot \|e\| \text{ for } g \in X^\# \text{ and } e \in F.$$

$$(\widetilde{\text{QNOI}}_1) \quad \text{The } p\text{-triangle inequality holds:}$$

$$\mathbf{A}^L(S + T)^p \leq \mathbf{A}^L(S)^p + \mathbf{A}^L(T)^p \text{ for } S, T \in \mathfrak{A}^L(X, F).$$

$$(\widetilde{\text{QNOI}}_2) \quad \mathbf{A}^L(BT A) \leq \|B\| \mathbf{A}^L(T) \text{Lip}(A) \text{ for } A \in \text{Lip}(X_0, X), T \in \mathfrak{A}^L(X, F), \text{ and } B \in \mathfrak{L}(F, F_0).$$

Remark 16. • *A **Lipschitz p-Banach nonlinear ideal** $[\mathfrak{A}^L, \mathbf{A}^L]$ is a nonlinear ideal \mathfrak{A}^L with a Lipschitz p-norm \mathbf{A}^L such that all linear spaces $\mathfrak{A}^L(X, F)$ are complete.*

- *We call \mathbf{A}^L a **strongly nonlinear ideal norm** if the condition $(\widetilde{\text{QNOI}}_0)$ is replaced by:*

$$\mathbf{A}^L(g \boxtimes e) \leq \text{Lip}(g) \cdot \|e\| \quad \text{for } g \in X^\# \text{ and } e \in F.$$

Proposition 25. Let $[\mathfrak{A}^L, \mathbf{A}^L]$ be a Lipschitz p -normed nonlinear ideal. Then $Lip(T) \leq \mathbf{A}^L(T)$ for all $T \in \mathfrak{A}^L$.

Proof. Let T be an arbitrary Lipschitz operator in $\mathfrak{A}^L(X, F)$.

$$\begin{aligned} Lip(T) &= \left\| T|_{F^*}^\# \right\| = \sup \left\{ Lip(T^\# b^*) : b^* \in B_{F^*} \right\} = \sup \left\{ Lip(b^* \circ T) : b^* \in B_{F^*} \right\} \\ &= \sup \left\{ \mathbf{A}^L((b^* \circ T) \boxtimes 1) : b^* \in B_{F^*} \right\} \\ &= \sup \left\{ \mathbf{A}^L(b^* \circ T) : b^* \in B_{F^*} \right\} \\ &\leq \mathbf{A}^L(T). \end{aligned}$$

■

Remark 17. If $p = 1$, then \mathbf{A}^L is simply called a **Lipschitz norm** and $[\mathfrak{A}^L, \mathbf{A}^L]$ is said to be a **Lipschitz Banach nonlinear ideal**.

We now formulate an important criterion which will be permanently used in the sequel.

Auto12

Proposition 26. Let \mathfrak{A}^L be a subclass of Lip with an \mathbb{R}^+ -valued function \mathbf{A}^L such that the following conditions are satisfied ($0 < p \leq 1$):

- (1) $g \boxtimes e \in \mathfrak{A}^L(X, F)$ and $\mathbf{A}^L(g \boxtimes e) \leq Lip(g) \cdot \|e\|$ for $g \in X^\#$ and $e \in F$.
- (2) It follows from $S_1, S_2, S_3, \dots \in \mathfrak{A}^L(X, F)$ and $\sum_{n=1}^{\infty} \mathbf{A}^L(S_n)^p < \infty$ that $S := \sum_{n=1}^{\infty} S_n \in \mathfrak{A}^L(X, F)$ and $\mathbf{A}^L(S)^p \leq \sum_{n=1}^{\infty} \mathbf{A}^L(S_n)^p$.
- (3) $A \in Lip(X_0, X)$, $T \in \mathfrak{A}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$ imply $BT A \in \mathfrak{A}^L(X_0, F_0)$ and $\mathbf{A}^L(BT A) \leq \|B\| \mathbf{A}^L(T) Lip(A)$.

Then $[\mathfrak{A}^L, \mathbf{A}^L]$ is a strongly Lipschitz p -Banach nonlinear ideal.

Proof. The only point is to observe that (2) summarizes the p -triangle inequality and the completeness, as well.

■

4.1 Lipschitz Nuclear Operators

A Lipschitz operator $T \in Lip(X, F)$ is called Lipschitz nuclear if

$$T = \sum_{j=1}^{\infty} g_j \boxtimes e_j,$$

with $g_1, g_2, g_3, \dots \in X^\#$ and $e_1, e_2, e_3, \dots \in F$ such that $\sum_{j=1}^{\infty} Lip(g_j) \|e_j\| < \infty$.

We put

$$\mathbf{N}^L(T) := \inf \sum_{j=1}^{\infty} \text{Lip}(g_j) \|e_j\|,$$

where the infimum is taken over all so-called Lipschitz nuclear representations described above.

The class of all Lipschitz nuclear operators is denoted by \mathfrak{N}^L .

Remark 18. *The series $\sum_{j=1}^{\infty} g_j \boxtimes e_j$ converges in the Lipschitz norm topology of $\text{Lip}(X, F)$. Therefore Lipschitz nuclear operators can be approximated by Lipschitz finite rank operators.*

Proposition 27. $[\mathfrak{N}^L, \mathbf{N}^L]$ is a strongly Lipschitz normed nonlinear ideal.

Proof. We use criterion 26.

- (1) Let $g \in X^\#$ and $e \in F$, it follows that $g \boxtimes e \in \mathfrak{N}^L(X, F)$ and $\mathbf{N}^L(g \boxtimes e) \leq \text{Lip}(g) \cdot \|e\|$.
- (2) Let $T_1, T_2, T_3, \dots \in \mathfrak{N}^L(X, F)$ such that $\sum_{n=1}^{\infty} \mathbf{N}^L(T_n) < \infty$. Given $\epsilon > 0$, choose nuclear representations $T_n = \sum_{j=1}^{\infty} g_{nj} \boxtimes e_{nj}$ with $\sum_{j=1}^{\infty} \text{Lip}(g_{nj}) \|e_{nj}\| \leq (1 + \epsilon) \mathbf{N}^L(T_n)$. Then $T := \sum_{n=1}^{\infty} T_n = \sum_{n,j=1}^{\infty} g_{nj} \boxtimes e_{nj}$ and $\sum_{n,j=1}^{\infty} \text{Lip}(g_{nj}) \|e_{nj}\| \leq (1 + \epsilon) \sum_{n=1}^{\infty} \mathbf{N}^L(T_n)$ imply $T \in \mathfrak{N}^L(X, F)$ and $\mathbf{N}^L(T) \leq (1 + \epsilon) \sum_{n=1}^{\infty} \mathbf{N}^L(T_n)$.
- (3) Let $T \in \mathfrak{N}^L(X, F)$ and $\epsilon > 0$. Consider a nuclear representation $T = \sum_{j=1}^{\infty} g_j \boxtimes e_j$ such that $\sum_{j=1}^{\infty} \text{Lip}(g_j) \|e_j\| \leq (1 + \epsilon) \mathbf{N}^L(T)$. If $A \in \text{Lip}(X_0, X)$ and $B \in \mathfrak{L}(F, F_0)$, then $BTA = \sum_{j=1}^{\infty} T^\# g_j \boxtimes B e_j$ and $\sum_{j=1}^{\infty} \text{Lip}(T^\# g_j) \|B e_j\| \leq (1 + \epsilon) \|B\| \mathbf{N}^L(T) \text{Lip}(A)$. This proves that $BTA \in \mathfrak{N}^L(X_0, F_0)$ and $\mathbf{N}^L(BTA) \leq (1 + \epsilon) \|B\| \mathbf{N}^L(T) \text{Lip}(A)$.

■

4.2 Lipschitz Hilbert Operators

A Lipschitz operator $T \in \text{Lip}(X, F)$ is called Lipschitz Hilbert operator if $T = BA$ with $A \in \text{Lip}(X, H)$ and $B \in \mathfrak{L}(H, F)$, where H is a Hilbert space.

We put

$$\mathbf{H}^L(T) := \inf \|B\| \text{Lip}(A),$$

where the infimum is taken over all possible factorizations.

The class of all Lipschitz Hilbert operators is denoted by \mathfrak{H}^L .

Proposition 28. $[\mathfrak{H}^L, \mathbf{H}^L]$ is a strongly Lipschitz normed nonlinear ideal.

Proof. We use criterion 26.

- (1) Let $g \in X^\#$ and $e \in F$, since \mathbb{K} is a Hilbert space, we have $g \boxtimes e = (1 \otimes e) \circ (g \boxtimes 1) \in \mathfrak{H}^L(X, F)$, where $1 \otimes e \in \mathfrak{L}(\mathbb{K}, F)$ and $g \boxtimes 1 \in Lip(X, \mathbb{K})$ and $\mathbf{H}^L(g \boxtimes e) \leq Lip(g) \cdot \|e\|$.
- (2) Let $T_1, T_2, T_3, \dots \in \mathfrak{H}^L(X, F)$ such that $\sum_{n=1}^{\infty} \mathbf{H}^L(T_n) < \infty$. Given $\epsilon > 0$, we choose factorizations $T_n = B_n A_n$ such that $A_n \in Lip(X, H_n)$ and $B_n \in \mathfrak{L}(H_n, F)$ satisfy the conditions

$$Lip(A_n)^2 \leq (1 + \epsilon) \mathbf{H}^L(T_n) \quad \text{and} \quad \|A_n\|^2 \leq (1 + \epsilon) \mathbf{H}^L(T_n).$$

From the Hilbert space $H := \ell_2(H_n)$. Put

$$A := \sum_{n=1}^{\infty} J_n A_n \quad \text{and} \quad B := \sum_{n=1}^{\infty} B_n Q_n.$$

Then

$$Lip(A)^2 \leq \sum_{n=1}^{\infty} Lip(A_n)^2 \quad \text{and} \quad \|B\|^2 \leq \sum_{n=1}^{\infty} \|B_n\|^2.$$

Finally, it follows from $T = BA$ that $T \in \mathfrak{H}^L(X, F)$. Moreover, we have

$$\mathbf{H}^L(T) \leq \|B\| Lip(A) \leq (1 + \epsilon) \sum_{n=1}^{\infty} \mathbf{H}^L(T_n).$$

- (3) This property is trivial. ■

4.3 Products of Lipschitz r -Normed Nonlinear Ideals

Let $[\mathfrak{A}, \mathbf{A}]$ and $[\mathfrak{A}^L, \mathbf{A}^L]$ be p -normed ideal and Lipschitz q -normed nonlinear ideal, respectively. For every Lipschitz operator $T \in Lip(X, F)$ belonging to the product $\mathfrak{A} \circ \mathfrak{A}^L$ we put

$$\mathbf{A} \circ \mathbf{A}^L(T) := \inf \mathbf{A}(B) \mathbf{A}^L(A),$$

where the infimum is taken over all factorizations $T = B \circ A$ with $B \in \mathfrak{A}(M, F)$ and $A \in \mathfrak{A}^L(X, M)$ and $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$.

Remark 19. The product $[\mathfrak{A} \circ \mathfrak{A}^L, \mathbf{A} \circ \mathbf{A}^L]$ will frequently be written as $[\mathfrak{A}, \mathbf{A}] \circ [\mathfrak{A}^L, \mathbf{A}^L]$.

Proposition 29. $[\mathfrak{A} \circ \mathfrak{A}^L, \mathbf{A} \circ \mathbf{A}^L]$ is strongly Lipschitz r -Banach nonlinear ideal.

Proof. From Proposition 4 we have $\mathfrak{A} \circ \mathfrak{A}^L$ is nonlinear ideal. The condition $(\widetilde{\mathbf{QNOI}}_0)$ satisfied. To prove condition $(\widetilde{\mathbf{QNOI}}_1)$, let T_1 and T_2 in $\mathfrak{A} \circ \mathfrak{A}^L(X, F)$. Given $\epsilon > 0$. Then there exists Banach spaces M_1, M_2 and operators $B_i \in \mathfrak{A}(M_i, F)$ and $A_i \in \mathfrak{A}^L(X, M_i)$ so that $T_i = B_i \circ A_i$ with

$$\mathbf{A}(B_1) \leq [(1 + \epsilon) \cdot \mathbf{A} \circ \mathbf{A}^L(T_1)]^{\frac{r}{p}},$$

$$\mathbf{A}(B_2) \leq [(1 + \epsilon) \cdot \mathbf{A} \circ \mathbf{A}^L(T_2)]^{\frac{r}{p}},$$

$$\mathbf{A}^L(A_1) \leq [(1 + \epsilon) \cdot \mathbf{A} \circ \mathbf{A}^L(T_1)]^{\frac{r}{q}},$$

$$\mathbf{A}^L(A_2) \leq [(1 + \epsilon) \cdot \mathbf{A} \circ \mathbf{A}^L(T_2)]^{\frac{r}{q}}.$$

Therefore the construction of condition $(\widetilde{\mathbf{NOI}}_1)$ in Proposition 4 implies that

$$\begin{aligned} \mathbf{A} \circ \mathbf{A}^L(T_1 + T_2)^r &= \mathbf{A} \circ \mathbf{A}^L(B \circ A)^r \leq \mathbf{A}(B)^r \mathbf{A}^L(A)^r \\ &\leq [\mathbf{A}(B_1)^p + \mathbf{A}(B_2)^p]^{\frac{r}{p}} [\mathbf{A}^L(A_1)^q + \mathbf{A}^L(A_2)^q]^{\frac{r}{q}} \\ &\leq [(1 + \epsilon)^r (\mathbf{A} \circ \mathbf{A}^L(T_1)^r + \mathbf{A} \circ \mathbf{A}^L(T_2)^r)]^{\frac{r}{p}} [(1 + \epsilon)^r (\mathbf{A} \circ \mathbf{A}^L(T_1)^r + \mathbf{A} \circ \mathbf{A}^L(T_2)^r)]^{\frac{r}{q}} \\ &\leq (1 + \epsilon)^r (\mathbf{A} \circ \mathbf{A}^L(T_1)^r + \mathbf{A} \circ \mathbf{A}^L(T_2)^r). \end{aligned}$$

To prove condition $(\widetilde{\mathbf{QNOI}}_2)$, let $A \in Lip(X_0, X)$, $T \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$, and $B \in \mathfrak{L}(F, F_0)$. From the construction of condition $(\widetilde{\mathbf{NOI}}_2)$ in Proposition 4 we have

$$\begin{aligned} \mathbf{A} \circ \mathbf{A}^L(BTA) &= \mathbf{A} \circ \mathbf{A}^L(\widetilde{\widetilde{B}} \circ \widetilde{\widetilde{A}}) := \inf \mathbf{A}(B \circ \widetilde{\widetilde{B}}) \mathbf{A}^L(\widetilde{\widetilde{A}} \circ A) \\ &\leq \mathbf{A}(B \circ \widetilde{\widetilde{B}}) \mathbf{A}^L(\widetilde{\widetilde{A}} \circ A) \\ &\leq \|B\| \mathbf{A}(\widetilde{\widetilde{B}}) \mathbf{A}^L(\widetilde{\widetilde{A}}) Lip(A) \\ &\leq \|B\| \mathbf{A} \circ \mathbf{A}^L(T) Lip(A). \end{aligned}$$

To show the completeness, let $T_n \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$ such that $\sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r < \infty$. Given $\epsilon > 0$, we can find factorizations $T = B_n \circ A_n$ such that the following conditions are satisfied:

$$B_n \in \mathfrak{A}(M_n, F) \quad \text{and} \quad \mathbf{A}(B_n) \leq [(1 + \epsilon) \mathbf{A} \circ \mathbf{A}^L(T_n)]^{\frac{r}{p}},$$

$$A_n \in \mathfrak{A}^L(X, M_n) \quad \text{and} \quad \mathbf{A}^L(A_n) \leq [(1 + \epsilon) \mathbf{A} \circ \mathbf{A}^L(T_n)]^{\frac{r}{q}}.$$

Put $B := \sum_{n=1}^{\infty} B_n Q_n$, $A := \sum_{n=1}^{\infty} J_n A_n$, and $M := \ell_2(M_n)$. Then

$$\sum_{n=1}^{\infty} \mathbf{A}(B_n)^p \leq (1 + \epsilon)^r \sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r,$$

and

$$\sum_{n=1}^{\infty} \mathbf{A}^L(A_n)^q \leq (1 + \epsilon)^r \sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r.$$

imply $B \in \mathfrak{A}(M, F)$ and $A \in \mathfrak{A}^L(X, M)$. Moreover, we have

$$\mathbf{A}(B)^p \leq (1 + \epsilon)^r \sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r,$$

and

$$\mathbf{A}^L(A)^q \leq (1 + \epsilon)^r \sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r.$$

Since $T := \sum_{n=1}^{\infty} T_n$ has the factorization $T = B \circ A$ it follows that $T \in \mathfrak{A} \circ \mathfrak{A}^L(X, F)$ and

$$\mathbf{A} \circ \mathbf{A}^L(T)^r \leq (1 + \epsilon)^r \sum_{n=1}^{\infty} \mathbf{A} \circ \mathbf{A}^L(T_n)^r.$$

Hence Proposition 26 yields the assertion. ■

4.4 Quotients of Lipschitz r -Normed Nonlinear Ideals

Let $[\mathfrak{A}, \mathbf{A}]$ and $[\mathfrak{A}^L, \mathbf{A}^L]$ be p -normed ideal and Lipschitz q -normed nonlinear ideal, respectively. For every Lipschitz operator $T \in Lip(X, F)$ belonging to the left-hand quotient $\mathfrak{A}^{-1} \circ \mathfrak{A}^L$ we put

$$\mathbf{A}^{-1} \circ \mathbf{A}^L(T) := \sup \{ \mathbf{A}^L(B \circ T) : B \in \mathfrak{A}(F, F_0), \mathbf{A}(B) \leq 1 \},$$

where F_0 is an arbitrary Banach space and $\frac{1}{r} := \frac{1}{p} + \frac{1}{q}$.

Remark 20. The quotient $[\mathfrak{A}^{-1} \circ \mathfrak{A}^L, \mathbf{A}^{-1} \circ \mathbf{A}^L]$ will frequently be written as $[\mathfrak{A}, \mathbf{A}]^{-1} \circ [\mathfrak{A}^L, \mathbf{A}^L]$.

Proposition 30. $[\mathfrak{A}^{-1} \circ \mathfrak{A}^L, \mathbf{A}^{-1} \circ \mathbf{A}^L]$ is strongly Lipschitz r -Banach nonlinear ideal.

Proof. The main point is to establish the existence of $\mathbf{A}^{-1} \circ \mathbf{A}^L$. Therefore let us suppose that the supremum is not finite for some Lipschitz operator $T \in \mathfrak{A}^{-1} \circ \mathfrak{A}^L(X, F)$. Then we can find $B_n \in \mathfrak{A}(F, F_n)$ such that

$$\mathbf{A}(B_n) \leq (2)^{-n} \quad \text{and} \quad \mathbf{A}^L(B_n \circ T) \geq n \quad \text{for} \quad n = 1, 2, 3, \dots$$

Put $F_0 := \ell_2(F_n)$. Since

$$\mathbf{A} \left(\sum_{n=h+1}^k J_n B_n \right) \leq \sum_{j=1}^{\infty} \mathbf{A}(B_{h+j}) \leq (2)^{-h},$$

the partial sums $\sum_{n=1}^k J_n B_n$ form an \mathbf{A} -Cauchy sequence. Consequently $B := \sum_{n=1}^{\infty} J_n B_n$ belongs to \mathfrak{A} , and we obtain $n \leq \mathbf{A}^L(B_n \circ T) = \mathbf{A}^L(Q_n B T) \leq \mathbf{A}^L(B T)$, which is a contradiction.

Finally, it is easy to check the nonlinear ideal properties and the completeness, as well. ■

References

- [1] M. G. Cabrera–Padilla, J. A. Chávez-Domínguez, A. Jimenez–Vargas and M. Villegas–Vallecillos, *Lipschitz Tensor Product*, submitted.
- [2] A. Pietsch, *Operator Ideals*, Deutsch. Verlag Wiss., Berlin, 1978; North–Holland, Amsterdam–London–New York–Tokyo, 1980.
- [3] A. Pietsch, *History of Banach Spaces and Linear Operators*, Birkhäuser Boston, 2007; North–Holland, Amsterdam–London–New York–Tokyo, 1980.
- [4] A. Jiménez-Vargas, J. M. Sepulcre, and Moisés Villegas-Vallecillos, *Lipschitz compact operators*, J. Math. Anal. Appl., 415 (2014), no. 2, 889–901.
- [5] A. Pietsch, *Eigenvalues and s -Numbers*, Geest & Portig, Leipzig, and Cambridge Univ. Press, 1987.
- [6] M. A. S. Saleh, *New types of Lipschitz summing maps between metric spaces*, Mathematische Nachrichten.